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A NOTE ON EXTREMAL METRICS OF NON-CONSTANT SCALAR CURVATURE

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ABSTRACT

By working in $\mathbb{C}^n$ with potentials of the form $a \log u + s(u)$, u the square of the distant to the origin, we obtain extremal Kähler metrics of non-constant scalar curvature on the blow-up of $\mathbb{C}^n$ at $\vec{0}$. We then show that these metrics can be completed at ∞ by adding a $\mathbb{CP}^{n-1}$, and reobtain the extremal Kähler metrics of non-constant scalar curvature constructed by Calabi on the blow-up of $\mathbb{CP}^n$ at one point. A similar construction produces this type of metrics on other bundles over $\mathbb{CP}^{n-1}$.

Let (M, J) be a complex n-manifold and g a *Hermitian* Riemannian metric on M, that is to say, a metric such that $\omega(X,Y) := g(JX,Y)$ is skew-symmetric. The metric is said to be *Kähler* if the 2-form ω is closed, or equivalently, if the almost-complex structure tensor J is parallel with respect to the metric connection ∇. The differential form ω is then called the *Kähler form*, and its cohomology class $[\omega] \in H^2_{DR}(M)$ is called the *Kähler class*.

By complex multi-linearity, we may extend the metric g, the Levi-Civita connection ∇ and the curvature tensor $\mathcal{R}$ to the complexified tangent bundle $\mathbb{C} \otimes TM$. Since $\mathbb{C} \otimes TM$ decomposes into the $\pm i$-eigenspaces of J, $\mathbb{C} \otimes TM = T^{1,0}M \oplus T^{0,1}M$, we can express any tensor field or differential operator in terms of the corresponding decomposition. If $(z^1, \ldots, z^n)$ is a holomorphic coordinate system on M, we get induced bases $\{\frac{\partial}{\partial z^j}\}$ and $\{\frac{\partial}{\partial z^{\bar{j}}} := \frac{\partial}{\partial \bar{z}^j}\}$ for $T^{1,0}M$ and $T^{0,1}M$, respectively, and if, for example, we express the metric g in terms of

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this basis by setting $g_{\mu\nu} := g\left(\frac{\partial}{\partial z^{\mu}}, \frac{\partial}{\partial z^{\nu}}\right)$, where the indices μ, ν range over $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$, it follows from the Hermiticity condition that $g_{jk} = g_{\bar{j}\bar{k}} = 0$, and that $\omega = \omega_{j\bar{k}} dz^j \wedge d\bar{z}^k = i g_{j\bar{k}} dz^j \wedge d\bar{z}^k$.

The *Ricci form* ρ is defined in terms of the Ricci tensor $\mathbf{r}$ of g by $\rho(X, Y) = \mathbf{r}(JX, Y)$. It is the curvature of the anti-canonical line bundle $\kappa^{-1} := \bigwedge^n T^{1,0}M$, so that the components of the Ricci tensor are given by

$$\mathbf{r}_{j\bar{k}} = -i\rho_{j\bar{k}} = -\frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log\det(g_{p\bar{q}}) \,.$$

The *scalar curvature* is, by definition, the trace $\sigma = \mathbf{r}^{\mu}_{\mu} = 2g^{j\bar{k}}\mathbf{r}_{j\bar{k}}$ of the Ricci tensor, and can be conveniently calculated by using the formula

$$\sigma\, \omega^{\wedge n} = 2n\ \rho \wedge \omega^{\wedge(n-1)}\,, \tag{1}$$

which follows from the Lefschetz decomposition of (1,1)-forms. For M compact, using the fact that the metric volume-form is given by $d\mathbf{vol}_g = \frac{\omega^{\wedge n}}{n!}$, this has the remarkable consequence that

$$\int_M \sigma\ d\mathbf{vol}_g = \frac{4\pi}{(n-1)!}\mathbf{c}_1 \cup [\omega]^{\cup(n-1)}.$$

Since, on the other hand, $\int_M d\mathbf{vol}_g = \frac{1}{n!}[\omega]^{\cup n}$, we conclude that *the average scalar curvature is a topological invariant*, in the sense that it only depends on the Kähler class $[\omega]$ and the homotopy class of the complex-structure tensor J.

Let $\mathcal{M}_{[\omega]}$ be the set of all Kähler metrics on M with fixed Kähler class $[\omega]$. Two metrics in this space will have Kähler forms which differ from one another by $i\partial\bar{\partial}\varphi$, where φ is a real valued function on M. This clearly indicates how to topologize $\mathcal{M}_{[\omega]}$. A curve ω^t in this space, with $\omega^0 = \omega$, is thus given by

$$\omega^t = \omega_{i\bar{j}} + i\partial_i\partial_{\bar{j}}\varphi_t\,, \tag{2}$$

where φ_t is a smooth valued function which can be taken as zero when $t = 0$.

The notion of extremal metric was introduced by Calabi [Ca2] as critical points of the functional

$$\begin{array}{c} \mathcal{M}_{[\omega]} \xrightarrow{S} \mathbb{R} \\ g \longrightarrow \int_M \sigma_g^2 d\mathbf{vol}_g \end{array}\,.$$

If we consider a one-parameter family of metrics as in (2) with $\varphi_t \mid_{t=0}= 0$ and $\dot{\varphi}_t \mid_{t=0}= u$, since the variation of σ_g is equal to

$$-\frac{1}{2}(\Delta^2 + 2\mathbf{r} \cdot \nabla\nabla)u\,,$$

where the dot denotes contraction, it follows, using the Bianchi identities, that the derivative of $\mathcal{S}$ at g in the direction of $\partial_i\partial_{\bar{j}}u$ is just

$$D\mathcal{S}_g(\partial_i\partial_{\bar{j}}u) = \int_M \sigma_g((\overline{\partial}\partial^{\#})^*\overline{\partial}\partial^{\#})ud\mathbf{vol}_g\,,$$

where $\partial^{\#}$ is the operator $\phi \mapsto (\overline{\partial}\phi)^{\#}$, # the isomorphism of the complexified cotangent and tangent bundles induced by g. The following result follows.

THEOREM 1 (Calabi [Ca2]): *A Kähler metric is extremal iff the vector field $\partial^{\#}\sigma_g$ is holomorphic.*

It is then rather obvious that a constant scalar curvature Kähler metric is extremal and that the two type of metrics coincide if the manifold admits no holomorphic vector fields.

We would like now to show examples of extremal metrics which do not have constant scalar curvature. Such examples have been constructed by Calabi [Ca1] and live in the blow-up of $\mathbb{CP}^n$ at one point. We hope that our approach illuminates further Calabi's construction, as some of the technicalities of his construction are eliminated in the approach that we follow here.

We initially search for this kind of metrics on the non-compact manifold $\mathbb{C}^n$ and demand that they be $\mathbf{U}(n)$ symmetric. We then proceed as in [Si] and consider a Kähler potential ϕ in $\mathbb{C}^n$ which is only a function of $u = z_1\bar{z}_1 + \ldots + z_n\bar{z}_n$. The $(1,1)$-form

$$\omega = i\partial\overline{\partial}\phi$$

is the Kähler form of a Kähler metric. If we let the expression

$$\omega^{\wedge n} = i^n V dz^1 \wedge d\overline{z}^1 \wedge dz^2 \wedge d\overline{z}^2 \wedge \ldots \wedge dz^n \wedge d\overline{z}^n$$

define the function V, then the Ricci form ρ is given by

$$\rho = -i\partial\overline{\partial}\log V\,.$$

The symmetry of our setting permits to carry the computations on $z^2 = z^3 = \ldots = z^n = 0$, and as explained in [Si]* the scalar curvature equation (1) is just

$$\sigma\dot{\phi}(\dot{\phi} + u\ddot{\phi}) = -2\dot{\phi}(\dot{\psi} + u\ddot{\psi}) - 2(n-1)\dot{\psi}(\dot{\phi} + u\ddot{\phi})\,, \tag{3}$$

where $\psi = \log V$. This is clearly equivalent to

$$\dot{\sigma} = \frac{1}{(u\dot{\phi})^n}\frac{d}{du}\left(\sigma u^n\dot{\phi}^n + 2nu^n\dot{\phi}^{n-1}\dot{\psi}\right).$$

Consequently, the vector field $\partial^{\#}\sigma$ is

$$\partial^{\#}\sigma = \sum_j \frac{\dot{\sigma}z^j}{\dot{\phi} + u\ddot{\phi}}\frac{\partial}{\partial z^j} = \sum_j \frac{z^j}{\dot{\phi} + u\ddot{\phi}}\frac{1}{(u\dot{\phi})^n}\frac{d}{du}\left(\sigma u^n\dot{\phi}^n + 2nu^n\dot{\phi}^{n-1}\dot{\psi}\right)\frac{\partial}{\partial z^j}.$$

The metric will be extremal iff

$$\frac{d}{du}\frac{\dot{\sigma}}{\dot{\phi} + u\ddot{\phi}} = 0,$$

from which it follows that

$$\frac{d}{du}\sigma = c\frac{d}{du}(u\dot{\phi}), \tag{4}$$

for some constant c. The case $c = 0$ was considered in [Si], taking advantage of the formula above for $\dot{\sigma}$. We concentrate here in the case where $c \neq 0$, i.e., the case of non-constant scalar curvature. Integration of (4), together with the use of (3), results into the equation

$$2\frac{d}{du}(u\dot{\psi}) + 2(n-1)\frac{\dot{\psi}}{\dot{\phi}}\frac{d}{du}(u\dot{\phi}) + c(u\dot{\phi})\frac{d}{du}(u\dot{\phi}) = -A\frac{d}{du}(u\dot{\phi}), \tag{5}$$

for A a constant of integration. This equation can be written as

$$2\frac{d}{du}(\zeta^{n-1}\eta) + c\zeta^n\dot{\zeta} = -A\zeta^{n-1}\dot{\zeta},$$

where $\zeta = u\dot{\phi}$, $\eta = u\dot{\psi}$, respectively. Thus,

$$\eta + \frac{c}{2(n+1)}\zeta^2 = -\frac{A}{2n}\zeta + B\zeta^{1-n}, \tag{6}$$

where B is another constant of integration. If we now write V in terms of ζ and its derivative, we conclude that

$$\eta = u\left(\frac{\ddot{\zeta}}{\dot{\zeta}} + \frac{(n-1)\dot{\zeta}}{\zeta} - \frac{n-1}{u}\right).$$

* The conventions used here are slightly different than the ones in [Si], thus explaining the differences in the equations.

Plugging this into the previous expression and writing the result as an equation for ζ, we obtain

$$\frac{d}{du}\left(u\zeta^{n-1}\dot{\zeta}\right) - n\zeta^{n-1}\dot{\zeta} + \frac{c}{2(n+1)}\zeta^{n+1}\dot{\zeta} = -\frac{A}{2n}\zeta^{n}\dot{\zeta} + B\dot{\zeta}\,,$$

which itself produces

$$u\zeta^{n-1}\dot{\zeta} - \zeta^{n} + \frac{c}{2(n+1)(n+2)}\zeta^{n+2} = -\frac{A}{2n(n+1)}\zeta^{n+1} + B\zeta + C\,, \tag{7}$$

for yet another constant of integration C.

In order to produce metrics in the blow-up of $\mathbb{CP}^n$ at one point, we suppose that ϕ is a function of the form

$$\phi(u) = a\log u + s(u)\,,\ a > 0\,, \tag{8}$$

where in principle we only required $s(u)$ to be C^2 (later on we shall see that equation (7) on such a function will imply that s is smooth). This will produce a metric on the blow-up of $\mathbb{C}^n$ at $\vec{0}$, a coordinate neighborhood of the manifold in question. Notice that $V = n!\dot{\phi}^{n-1}(\dot{\phi} + u\ddot{\phi})$. Then, if we set

$$\dot{s}(0) = p\,,\ \ddot{s}(0) = q\,,$$

the constants A, B and C are completely determined by (5), (6), and (7) in terms of these values. We list these results for convenience:

$$\begin{cases} A = \dfrac{2(n-1)(n-2)}{a} - \dfrac{4q}{p^2} - ca\,, \\ B = -\dfrac{a^{n-1}}{n}\left(2(n-1) + \dfrac{2q}{p^2}a + \dfrac{c}{2(n+1)}a^2\right), \\ C = \dfrac{a^{n}}{n+1}\left(2(n-2) + \dfrac{2q}{p^2}a + \dfrac{c}{2(n+2)}a^2\right). \end{cases} \tag{9}$$

For ϕ as in (8) we have $\zeta = u\dot{\phi} = a + u\dot{s}$. If we write equation (7) in terms of s and perform some simplifications, we obtain

$$\begin{aligned}(a+u\dot{s})^{n-1}u^2\ddot{s} = &-\frac{c}{2(n+1)(n+2)}((a+u\dot{s})^{n+2} - a^{n+2} - (n+2)a^{n+1}u\dot{s}) \\ &-\frac{A}{2n(n+1)}((a+u\dot{s})^{n+1} - a^{n+1} - (n+1)a^{n}u\dot{s}) \\ &+((a+u\dot{s})^{n} - a^{n} - na^{n-1}u\dot{s}) - u\dot{s}((a+u\dot{s})^{n-1} - a^{n-1})\,.\end{aligned}$$

Setting $w = \dot{s}$ this becomes

$$
\begin{aligned}
\dot{w} = & -\frac{c((a+uw)^{n+2} - a^{n+2} - (n+2)a^{n+1}uw)}{2(n+1)(n+2)u^2(a+uw)^{n-1}} \\
& -\frac{A((a+uw)^{n+1} - a^{n+1} - (n+1)a^{n}uw)}{2n(n+1)u^2(a+uw)^{n-1}} \\
& +\frac{a(a+uw)^{n-1} - a^{n} - (n-1)a^{n-1}uw}{u^2(a+uw)^{n-1}} \\
= & f(u,w),
\end{aligned} \tag{10}
$$

where we let the expression define the function $f(u,w)$.

THEOREM 2: *The blow-up of $\mathbb{CP}^n$ at one point carries extremal metrics of non-constant scalar curvature.*

Proof: Let $\mathbb{C}^n$ be an affine coordinate chart, and assume that the origin is the blown-up point. Proceeding as above, we search for a potential of the form (8) which will produce extremal metrics of non-constant scalar curvature. As the function $f(u,w)$ is smooth, the resulting equation (10) for $w = \dot{s}$ has a unique solution in a neighborhood of $u = 0$, provided one supplies an initial condition $w(0) = \dot{s}(0) = p$. There is in fact one such solution which is smooth and defined on a maximal domain $[0, \alpha)$. Integrating it we obtain s, and with this s, the potential (8) produces an extremal metric in some subset of the blow-up at one point of $\mathbb{CP}^n$ minus a $\mathbb{CP}^{n-1}$, so long as $w(0) = p > 0$. Notice that in the resulting differential equation, the parameter $q = \dot{w}(0)$ is free.

With a positive initial condition p, the solution to (10) remains positive for any u in $[0, \alpha)$. This is so because if $w(u)$ were zero at one positive value of u, both $w(u)$ and $\dot{w}(u)$ would be zero at that point and, by uniqueness of solutions of differential equations, $w(u)$ would have to be identically zero.

If the constant c is strictly positive, the solution $w(u)$ cannot go to $+\infty$ in finite time. In fact, as $w \to +\infty$ and for nonzero u, the function $f(u,w)$ is asymptotically equals to $-cuw^3/2(n+1)(n+2)$, which is a negative quantity. Thus, for $c > 0$ the function w is decreasing when its value is large and, consequently, $\alpha = \infty$. So, the solution to (10) exists for $u \in [0, \infty)$. Under this assumption, $\dot{\phi} + u\ddot{\phi}$ is never zero, because if that were the case at some u, say u_0, then $\zeta_0 = u_0\dot{\phi}(u_0)$ would be a root of the polynomial defined by the terms in equation (7) which do not involve $\dot{\zeta}$. By uniqueness of solutions to differential equations, $\zeta = \zeta_0$, a constant, which is a contradiction. Therefore, $\dot{\phi}+u\ddot{\phi} = \dot{s}+u\ddot{s} > 0$, since it is so when $u = 0$. Consequently, $V = n!\dot{\phi}^{n-1}(\dot{\phi}+u\ddot{\phi})$

is always positive, showing that the form $\omega = i\partial\bar{\partial}\phi$ is non-degenerate. Thus, the function s obtained from

$$s(u) = pu + \int_0^u w(\tau)d\tau$$

is such that (8), translated to $\mathbb{CP}^n$ via the coordinate system, defines a Kähler metric in $\mathbb{CP}^n - \mathbb{CP}^{n-1}$ blown-up at one point, a metric which is extremal and has non-constant scalar curvature.

In addition to proving that the derivative of $u\dot{\phi}$ is positive, we have shown that this function is bounded on $[0,\infty)$. Hence, $\lim_{u\to\infty} u\dot{\phi} = b > a$, and equation (7) implies that $u(\dot{s} + u\ddot{s})$ converges to some limit as $u \to \infty$. This limit is necessarily zero.

For simplicity, let us set

$$P(\zeta) = -\frac{c}{2(n+1)(n+2)}\zeta^{n+2} - \frac{A}{2n(n+1)}\zeta^{n+1} + \zeta^n + B\zeta + C\,.$$

Then, (7) can be written as

$$u\zeta^{n-1}\dot{\zeta} = P(\zeta)\,.$$

Notice that $\zeta(0) = a$ and $\zeta(\infty) = b$ are both roots of $P(\zeta)$. Furthermore, they must be simple roots. The residue of $\zeta^{n-1}/P(\zeta)$ at $\zeta_0 = a$ or $\zeta_0 = b$ can be computed by the limit $\lim_{\zeta\to\zeta_0} \zeta^{n-1}/P'(\zeta)$. Given the value of the constants in (9), we see that the residue at $\zeta_0 = a$ is equal to 1. Later on we shall see that in order to extend the metric at $u = \infty$, we need to impose a condition on the residue at $\zeta_0 = b$.

We next give the metric explicitly near $u = \infty$. For that we introduce a new coordinate $r = \sqrt{\zeta}$ and choose a local orthonormal coframe $\sigma_1, \ldots, \sigma_{2n-1}$ for the sphere S^{2n-1} which coincides with $dx^2, dy^2, \ldots, dx^{2n-2}, dy^{2n-2}, dy^1$ at the point $(z^1, \ldots, z^n) = (1, 0, \ldots, 0)$. Here $z^k = x^k + iy^k$. Since

$$-i\omega = (\dot{\phi} + u\ddot{\phi})dz^1 \wedge d\bar{z}^1 + \dot{\phi}\sum_{j=2}^{n} dz^j \wedge d\bar{z}^j,$$

we obtain that

$$g = \frac{d\zeta}{du}\left(\frac{du^2}{4u} + u\sigma_{2n-1}^2\right) + r^2(\sigma_1^2 + \ldots + \sigma_{2n-2}^2)\,.$$

But

$$\frac{d\zeta}{du}\frac{du^2}{4u} = \frac{\zeta dr^2}{u\dot{\zeta}} = \frac{\zeta^n dr^2}{P(\zeta)} = \frac{r^{2n}}{P(r^2)}dr^2 .$$

Similarly,

$$u\dot{\zeta}\sigma_{2n-1}^2 = \frac{P(r^2)}{r^{2(n-1)}}\sigma_{2n-1}^2 .$$

Therefore,

$$g = \frac{r^{2n}}{P(r^2)}dr^2 + \frac{P(r^2)}{r^{2(n-1)}}\sigma_{2n-1}^2 + r^2(\sigma_1^2 + \ldots + \sigma_{2n-2}^2) .$$

The first two summands in the expression above, should be viewed as the fiber metric of a bundle whose base space has metric given by the last summand. This base space is a $\mathbb{CP}^{n-1}$ with a Fubini-Study metric, and as $r^2 \to b$ as $u \to \infty$, the scalar curvature is $4n(n-1)/b$. However, before we can safely conclude that, we must show that the metric extends smoothly at ∞. For that, we would like to find a new coordinate $v = \beta(r)$ such that the fiber metric above looks like

$$(1 + O(v^2))dv^2 + v^2(1 + O(v^2))\sigma_{2n-2}^2 ,$$

near $r^2 = b$. Let a_0 be the residue of $\zeta^{n-1}/P(\zeta)$ at $\zeta_0 = b$. Since

$$\frac{r^{2(n-1)}}{P(r^2)} = \frac{a_0}{r^2 - b} + Q(r^2 - b) ,$$

with Q a regular function near b, we can choose

$$\beta'(r) = -a_0 \frac{r}{\sqrt{b - r^2}} ,$$

which implies that $v = \beta(r) = a_0\sqrt{b - r^2}$. Since we then want $P(r^2)/r^{2(n-1)}$ to be of the form $v^2(1 + O(v^2))$, we must have

$$a_0^2(b - r^2) = \frac{r^2 - b}{a_0} ,$$

which then implies that $a_0 = -1$. Hence, the constants in (9) must be related to c and b by

$$-b^{n-1} = -\frac{c}{2(n+1)}b^{n+1} - \frac{A}{2n}b^n + nb^{n-1} + B ,$$

from which we obtain

$$\frac{2q}{p^2} = \frac{n}{b^n - a^n}\left[\frac{c}{2}(\frac{b^{n+1} + a^{n+1}}{n+1} - \frac{ab^n}{n}) + \frac{n-1}{n}(\frac{(n-2)b^n}{a} + 2a^{n-1}) - (n+1)b^{n-1}\right] .$$

This permits to express the constants in (9) in terms of the cohomological data a and b, and the dimension n. Fixing these data fix also the free parameter q.

With this restriction on the residue at $\zeta_0 = b$, the metric constructed extends smoothly across $u = \infty$ (because it does so across $v = 0$ which is equivalent). Adding a $\mathbb{CP}^{n-1}$ at ∞, we complete the blow-up of $\mathbb{C}^n$ at $\vec{0}$ to obtain the manifold $\mathbb{CP}^n$, blown-up at one point, provided with a metric which is extremal and has non-constant scalar curvature. ∎

It is easy to see that the volume of the compact manifold above is just $2^n \mathrm{vol}(S^{2n-1})(b^n - a^n)/n$, where $\mathrm{vol}(S^{2n-1})$ is the volume of the $(2n-1)$-dimensional unit sphere.

If instead of searching for potentials as in (8), we look for potentials of the form

$$\phi(u) = a \log u + u^k s(u)\,,\ a > 0\,,$$

with k a positive integer and $s(0) > 0$, we then obtain metrics with the properties above on L^k. Here, $L \xrightarrow{\pi} \mathbb{CP}^{n-1}$ is the universal line bundle obtained by blowing-up $\mathbb{C}^n$ at the origin (notice that the previous construction corresponds to $k = 1$). Adding a point at infinity on each fiber gives the result on a compact manifold. Details are left to the interested reader.

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